# The effect of a magnetic field on the flow of a conducting fluid past a body of revolution

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The problem described by the title is investigated when both the magnetic field and the streaming motion of the fluid at infinity are uniform and parallel to the axis of symmetry of the body. The flow pattern depends on three parameters, the Reynolds number $R$, the magnetic Reynolds number $R_m$ and the Hartmann number $M$. In this paper it is assumed that $M \gg 1$, $M \gg R$, $M \gg R_m$ (no other restrictions on the parameters are imposed, so that $R$ and $R_m$ need not be small). The flow pattern then consists of an undisturbed uniform stream outside a cylinder circumscribing the body with generators parallel to the stream. Inside this cylinder the fluid is at rest. The leading term in the expression for the drag on the body is obtained.

## 1. Introduction

In an earlier paper (Chester 1957) the author considered the effect of a magnetic field on the flow of a conducting fluid past a sphere. It was shown that the magnetic field, assumed to be parallel to the uniform stream at infinity, produces a body force which opposes the natural tendency of the fluid to flow round the body and thereby increase the drag. An expression for the drag was obtained as an expansion in increasing powers of the Hartmann number.

The object of the present investigation is to consider the same problem, but for large values of the Hartman number. Because of the insensitivity of the flow to the detailed shape of the body, it is possible to generalize the argument to deal with an arbitrary body of revolution.

## 2. Formulation of the problem

The fluid is assumed to be incompressible, viscous and conducting. It flows steadily past a body of revolution whose axis defines the $x$-axis of a system of Cartesian co-ordinates with origin inside the body. Both the flow direction and the magnetic field are parallel to the $x$-axis at infinity.

The equations to be solved are then, with the usual notation for electromagnetic quantities (measured in M.K.S. units),

$$\nabla' \wedge \mathbf{H} = \mathbf{j} = \sigma(\mathbf{E} + \mu \mathbf{V}' \wedge \mathbf{H}), \quad \nabla' . \mathbf{H} = 0, \quad \nabla' \wedge \mathbf{E} = 0, \tag{1}$$

$$\nabla' . \mathbf{V}' = 0, \tag{2}$$

$$\rho(\mathbf{V}' . \nabla')\mathbf{V}' = -\nabla' p' + \rho\nu\nabla'^2 \mathbf{V}' + \mu \mathbf{j} \wedge \mathbf{H}, \tag{3}$$

where $\mathbf{V}'$, $p'$, $\rho$, $\nu$ denote respectively the velocity, pressure, density and kinematic viscosity. The prime has been used here so that the same symbols without the prime can later be used to denote non-dimensional quantities.

We note first that, because of the symmetry of the problem, the vector $\mathbf{V}' \wedge \mathbf{H}$ has zero divergence and hence, by equations (1),

$$\nabla'.\mathbf{E} = 0, \quad \nabla' \wedge \mathbf{E} = 0. \tag{4}$$

The electric field is thus identically zero, since this is a solution of (4) which violates no boundary conditions at the body or at infinity.

Now let $U$ be the speed of the uniform stream at infinity, and let the body lie within a sphere of radius $a$ and centre at the origin. The space co-ordinates may then be made non-dimensional with the factor $a^{-1}$, and the pressure and velocity by the relations

$$p = \frac{a}{\rho\nu U}(p' - p'_\infty), \quad \mathbf{V} = \mathbf{V}'/U, \tag{5}$$

where $p'_\infty$ is the pressure at infinity.

It follows, from the first of equations (1), that

$$R_m \mathbf{V} \wedge \mathbf{H} = \nabla \wedge \mathbf{H}, \tag{6}$$

where $$R_m = Ua\mu\sigma = \text{magnetic Reynolds number}. \tag{7}$$

If the left-hand side of (6) were neglected, $\mathbf{H}$ would also satisfy equations (4). If, in addition, the permeabilities of the fluid and the body were equal, the magnetic field would be uniform and parallel to the $x$-axis. We thus write

$$\mathbf{H} = \pm H(\mathbf{i} + R_m \mathbf{A}), \tag{8}$$

where the first term is taken to be dominant. The validity of this assumption will be discussed later. For the moment we take $\mathbf{H} = \pm H\mathbf{i}$ approximately, and equation (3) then reduces to

$$R(\mathbf{V}.\nabla)\mathbf{V} = -\nabla p + \nabla^2 \mathbf{V} + M^2(\mathbf{V} \wedge \mathbf{i}) \wedge \mathbf{i}, \tag{9}$$

where $$R = Ua/\nu = \text{Reynolds number}, \tag{10}$$

$$M = \mu Ha(\sigma/\rho\nu)^{\frac{1}{2}} = \text{Hartmann number}. \tag{11}$$

Note that the Hartmann number, like the Reynolds number and magnetic Reynolds number, is essentially non-negative.

Equation (9) is further simplified by neglecting the term on the left-hand side. The techniques of this paper are not sufficiently refined to discuss the precise condition for which the neglect of the convective terms is valid. However, in the paper immediately following, a much more careful analysis of the whole flow field is made for the special case of a circular disk. There it is shown that the approximation is justified provided that $M \gg R$. (This is also the condition required when $M \ll 1$.) While this is not completely general it is also not so restrictive as might at first appear, for it is also found that the flow field is insensitive to the detailed shape of the body when $M$ is large. The uniform stream is undisturbed outside an elongated cylinder circumscribing the body; inside the cylinder the fluid is substantially at rest. The thickness of the shear layer at the surface of this cylinder is $O(M^{-\frac{1}{2}})$ near the body and gradually thickens towards far distant points, where the contour of the cylinder becomes less well defined and the flow merges into the pattern of a uniform stream at infinity.

## 3. General solution of the equations

With the indicated simplifications, (2) and (9) now give

$$\nabla . \mathbf{V} = 0, \tag{12}$$

$$-\nabla p + \nabla^2 \mathbf{V} + M^2 (\mathbf{V} \wedge \mathbf{i}) \wedge \mathbf{i} = 0. \tag{13}$$

These have the particular solution (Chester 1957)

$$\mathbf{V} = \mathbf{i} + e^{Mx} \nabla \phi_1 + e^{-Mx} \nabla \phi_2, \tag{14}$$

$$p = M e^{Mx} \frac{\partial \phi_1}{\partial x} - M e^{-Mx} \frac{\partial \phi_2}{\partial x}, \tag{15}$$

where

$$\nabla^2 \phi_1 + M \frac{\partial \phi_1}{\partial x} = 0, \tag{16}$$

$$\nabla^2 \phi_2 - M \frac{\partial \phi_2}{\partial x} = 0. \tag{17}$$

Equations (14) and (15) contain the most general solutions for $p$, and $u$, the $x$-component of $(\mathbf{V} - \mathbf{i})$. For it is easily deducible from (13) and (14) that

$$\nabla^2 (p + Mu) - M \frac{\partial}{\partial x} (p + Mu) = 0, \tag{18}$$

$$\nabla^2 (p - Mu) + M \frac{\partial}{\partial x} (p - Mu) = 0, \tag{19}$$

so that $(p + Mu)$ and $(p - Mu)$ can be identified with $Me^{Mx} \phi_{1x}$ and $-M e^{-Mx} \phi_{2x}$ respectively. Thus the most general solution may contain extra terms which contribute to the velocity components perpendicular to the $x$-axis. Since such terms must satisfy the continuity equation, they may be expressed in the form $\nabla \chi \wedge \mathbf{i}$. Since also the conditions of symmetry require that there be no component of vorticity in the $x$-direction, $\chi$ must be a solution of the two-dimensional Laplace's equation. This solution is to be regular at all points in a plane perpendicular to the $x$-axis and so can be at most a constant, which has no significance in the velocity field. Equations (14) and (15) therefore represent the complete solution.

It also follows, from (1), (6) and (8) that the equations satisfied by $\mathbf{A}$ outside the body are approximately

$$\nabla \wedge \mathbf{A} = \mathbf{V} \wedge \mathbf{i}, \quad \nabla . \mathbf{A} = 0, \tag{20}$$

with the solution

$$\mathbf{A} = -M^{-1} e^{Mx} \nabla \phi_1 + M^{-1} e^{-Mx} \nabla \phi_2 + \nabla \phi', \tag{21}$$

where

$$\nabla^2 \phi' = 0.$$

Inside the body, where $\nabla \wedge \mathbf{A} = 0$, $\nabla . \mathbf{A} = 0$, $\mathbf{A}$ is a harmonic function. This function, together with $\phi'$, is determined by the continuity of $\mathbf{A}$ on the surface of the body. Now the first two terms in (21) are at most $O(M^{-1})$ for, as will be shown, each term in (14) for the velocity is at most $O(1)$. It follows that $\mathbf{A}$ itself is at most $O(M^{-1})$ and hence that the approximation implied by taking the magnetic field to be $H\mathbf{i}$ is valid provided that $M \gg R_m$.

The assumptions made also imply that the Alfven wave speed $(\mu^{\frac{1}{2}}H/\rho^{\frac{1}{2}})$ is large compared with $U$, for their ratio is $M^2/RR_m$. The effect on disturbances associated with this speed may be compared to the effect on compressive disturbances of the assumption of small Mach number.

## 4. Solution for large Hartmann number

To obtain the solution for $M \gg 1$, it is sufficient to note that $p$ and each of the components of $(\mathbf{V}-\mathbf{i})$ can be written as the sum of two terms satisfying respectively the equations

$$\nabla^2\psi_1 - M\frac{\partial\psi_1}{\partial x} = 0, \tag{22}$$

$$\nabla^2\psi_2 + M\frac{\partial\psi_2}{\partial x} = 0. \tag{23}$$

The asymptotic behaviour of solutions of the first equation has already been discussed in the literature (Oseen 1927; Stewartson 1956) with reference to Oseen's equation for viscous flow past solid bodies, and the second equation is also amenable to similar treatment. We follow here the interpretation given by Stewartson (1956) of Oseen's original argument.

Let

$$\psi_1 = A_1(y',z'), \quad \frac{\partial\psi_1}{\partial\alpha} = MB_1(y',z'), \tag{24}$$

$$\psi_2 = A_2(y',z'), \quad \frac{\partial\psi_2}{\partial\alpha} = MB_2(y',z'), \tag{25}$$

at the point $(x',y',z')$ of the body, where $\partial/\partial\alpha$ represents differentiation along the outward normal. (It will appear later that the normal derivatives can be $O(M)$ on the body.) Then, by a simple extension of Kirchhoff's solution of the wave equation, we can write

$$4\pi\psi_1(x,y,z) = \iint\left\{M(B_1 - lA_1)\frac{e^{\frac{1}{2}M(x-x'-R)}}{R} - A_1\frac{\partial}{\partial\alpha}\left(\frac{e^{\frac{1}{2}M(x-x'-R)}}{R}\right)\right\}dS, \tag{26}$$

$$4\pi\psi_2(x,y,z) = \iint\left\{M(B_2 + lA_2)\frac{e^{-\frac{1}{2}M(x-x'+R)}}{R} - A_2\frac{\partial}{\partial\alpha}\left(\frac{e^{-\frac{1}{2}M(x-x'+R)}}{R}\right)\right\}dS, \tag{27}$$

where $(x',y',z')$ is a point on the body where the outward drawn normal has direction cosines $(l,m,n)$, and

$$R^2 = (x-x')^2 + (y-y')^2 + (z-z')^2.$$

Consider first the asymptotic behaviour of

$$\iint M(B_1 - lA_1)\frac{e^{\frac{1}{2}M(x-x'-R)}}{R}dS \tag{28}$$

for large $M$. The integrand is exponentially small unless $(y-y')^2 + (z-z')^2$ is small. This implies that the integral itself is exponentially small unless a line through $(x,y,z)$ parallel to the $x$-axis cuts the body. If it does cut the body at a point

$(\xi, y, z)$ then the sensible contribution to (28) comes from the integral over that part of the body in the neighbourhood of this point, together with similar contributions from the other points of intersection. The contribution from one point of intersection may then be written

$$M\{B_1(y,z) - lA_1(y,z)\} \iint \frac{e^{\frac{1}{2}M(x-x'-R)}}{R}\, dS, \tag{29}$$

where $l$ is also a function of $y$ and $z$, and the integration can be taken over the infinite plane $lx' + my' + nz' = l\xi + my + nz$.

With the transformation

$$x' - \xi = -(1-l^2)^{\frac{1}{2}} r \sin\theta + (1-l^2)(x-\xi),$$

$$y' - y = \frac{nr\cos\theta}{(1-l^2)^{\frac{1}{2}}} + \frac{mlr\sin\theta}{(1-l^2)^{\frac{1}{2}}} - lm(x-\xi),$$

$$z' - z = \frac{-mr\cos\theta}{(1-l^2)^{\frac{1}{2}}} + \frac{nlr\sin\theta}{(1-l^2)^{\frac{1}{2}}} - nl(x-\xi),$$

the integral (29) becomes

$$M(B_1 - lA_1)\int_0^\infty \frac{r\,dr}{\{r^2 + l^2(x-\xi)^2\}^{\frac{1}{2}}}\int_0^{2\pi} \exp \tfrac{1}{2}M[l^2(x-\xi) + (1-l^2)^{\frac{1}{2}} r\sin\theta - \{r^2 + l^2(x-\xi)^2\}^{\frac{1}{2}}]\, d\theta$$

$$= 2\pi M(B_1 - lA_1)\int_0^\infty \frac{r\,dr}{\{r^2 + l^2(x-\xi)^2\}^{\frac{1}{2}}} I_0\{\tfrac{1}{2}Mr(1-l^2)^{\frac{1}{2}}\} \times \exp \tfrac{1}{2}M[l^2(x-\xi) - \{r^2 + l^2(x-\xi)^2\}^{\frac{1}{2}}]$$

$$= \frac{4\pi(B_1 - lA_1)}{|l|} \exp[\tfrac{1}{2}Ml^2\{(x-\xi) - |x-\xi|\}]$$

$$= \begin{cases} 4\pi(B_1 - lA_1)/|l| & \text{for } x > \xi, \\ \dfrac{4\pi(B_1 - lA_1)}{|l|} \exp[Ml^2(x-\xi)] & \text{for } x < \xi. \end{cases} \tag{30}$$

Similarly, the contribution to

$$\iint A_1 \frac{\partial}{\partial \alpha} \frac{(e^{\frac{1}{2}M(x-x'-R)})}{R}\, dS$$

is sensibly

$$-A_1\left\{l\frac{\partial}{\partial x} + m\frac{\partial}{\partial y} + n\frac{\partial}{\partial z}\right\}\iint \frac{e^{\frac{1}{2}M(x-x'-R)}}{R}\, dS$$

$$= \begin{cases} 0 & \text{for } x > \xi, \\ -4\pi A_1 \operatorname{sgn} l\, e^{Ml^2(x-\xi)} & \text{for } x < \xi. \end{cases} \tag{31}$$

Similar contributions come from the other points of intersection. The final result is that $\psi_1$ tends exponentially to zero outside a cylinder circumscribing the body with its axis along the $x$-axis. Inside this cylinder on the downstream side of the body ($x > 0$) the asymptotic form of $\psi_1$ is independent of $x$ so that

$$\psi_1 = \psi_1(y,z),$$

and on the upstream side ($x < 0$)

$$\psi_1 = \beta(y,z)\, e^{Ml^2(x-\xi)},$$

where $\xi$ is the value of $x$ at the point on the body nearest the point $(x,y,z)$, for which $y = y'$, $z = z'$.

The argument for $\psi_2$ is similar and only the final results are quoted. Inside the circumscribing cylinder $\psi_2 = \psi_2(y,z)$ on the upstream side, and on the downstream side

$$\psi_2 = \beta'(y,z)\, e^{-Ml^2(x-\xi)}.$$

It is now a simple matter to write down the asymptotic forms of the velocity components and the pressure. If we consider only the perturbed field, all the components are asymptotically zero outside the circumscribing cylinder. Inside this cylinder on the downstream side of the body

$$M^{-1}p = u_0 - u_1 e^{-Ml^2(x-\xi)},$$

$$\mathbf{V} = \mathbf{i} + \mathbf{V}_0 + \mathbf{V}_1 e^{-Ml^2(x-\xi)},$$

where $u_0, u_1$ are the $x$-components respectively of $\mathbf{V}_0$, $\mathbf{V}_1$, these vectors being functions of $y$ and $z$ only. (Because of the symmetry of the flow they are in fact functions of $y^2+z^2$ only.) Since $\mathbf{V}$ satisfies the continuity equation (12), the same must also be true of $\mathbf{V}_0$ and $\mathbf{V}_1 \exp\{-Ml^2(x-\xi)\}$ separately. If $\mathbf{V}_0$ is to be regular on the axis it cannot satisfy both the continuity equation and the requirements imposed by symmetry unless $\mathbf{V}_0 \wedge \mathbf{i}$ is zero. Also on the body the boundary condition $\mathbf{V} \wedge \mathbf{i} = 0$ must be satisfied and this now implies $\mathbf{V}_1 \wedge \mathbf{i} = 0$. The continuity condition then requires that $\mathbf{V}_1$ itself be zero. Finally the remaining boundary condition at the body (on the $x$-component of the velocity) is satisfied if $u_0 = -1$.

Similar arguments apply on the upstream side of the body, the only difference being that the contribution from the terms corresponding to $\psi_1$ and $\psi_2$ are interchanged. Since the boundary conditions are unchanged, the velocity fields on the upstream and downstream sides of the body are identical. But, because of the interchange of the roles of $\psi_1$ and $\psi_2$, the pressures on the two sides of the body are equal in magnitude but of opposite sign.

The following description of the flow field now emerges.* Outside the circumscribing cylinder the uniform flow field parallel to the axis of the body continues unchecked. Inside the cylinder the fluid is stagnant, the non-dimensional pressure $p$ is equal to $M$ on the upstream side and $-M$ on the downstream side. The drag is clearly given by

$$\frac{D}{\rho\nu a U} = \frac{2MA}{a^2}$$

or

$$D = 2\mu H U (\sigma\rho\nu)^{\frac{1}{2}} A,$$

where $A$ is the cross-sectional area of the circumscribing cylinder. (Note that $D/\rho\nu a U$ and $A/a^2$ are the non-dimensional representations respectively of the drag and cross-sectional area corresponding to the pressure $p = ap'/\rho\nu U$.)

* The result is qualitatively similar to that of Stewartson (1960) for a similar problem in which the fluid was perfectly conducting and inviscid.

The discontinuity in velocity at the surface of the cylinder arises from the fact that the arguments used to obtain the asymptotic behaviour of $\psi_1$ and $\psi_2$ break down wherever $l$ (the $x$-component of the unit normal to the body) is zero. This discontinuity is discussed in greater detail in the following paper where it is shown that it is an approximation to a region of rapid transition which, however, becomes more diffuse as $x \to \pm \infty$. It will also be shown that this thickening of the shear layer at far distant points explains a further anomaly of the present analysis, namely that the boundary condition on the velocity at infinity is not satisfied inside the circumscribing cylinder. It is in fact more accurate to picture the cylinder as being of length $O(M)$ and the uniform stream being dominant everywhere at distances large compared with $O(M)$.

One further point requires comment. The above argument is sufficient to deal with the upstream and downstream sides of the body. However, it may happen that a line parallel to the $x$-axis cuts the body in more than two points so that a pocket of fluid appears in the vicinity of the body. If one such pocket is described by $\xi_1 < x < \xi_2$, then the asymptotic form of the velocity vector there will be

$$\mathbf{V} = \mathbf{i} + \mathbf{V}_0 + \mathbf{V}_1 e^{-Ml^2(x-\xi_1)} + \mathbf{V}_2 e^{Ml^2(x-\xi_2)}.$$

Moreover $\mathbf{V}_0$ will contain contributions from integrals over planes for which $x > \xi$ and $x < \xi$. These contributions to $u_0$ must now be known separately before an expression for the pressure can be deduced. The present theory does not predict this, and the pressure is indeterminate unless some further condition is invoked such as continuity at the boundary of the pocket. But as far as the drag is concerned, it is sufficient that the pressure be independent of $x$, and this is so since it is easily shown that $\mathbf{V}_0 \wedge \mathbf{i} = \mathbf{V}_1 = \mathbf{V}_2 = 0$.

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